

A POST-PROJECTIVE PART OF TILTING QUIVER OVER CERTAIN PATH ALGEBRAS

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ABSTRACT. D.Happel and L.Unger defined a partial order on set of basic tilting modules. A tilting quiver is a Hasse-diagram of this poset.

In this paper we see a structure of post-projective part of a tilting quiver over a path algebra satisfying some conditions.

INTRODUCTION

In this paper we use the following notations. Let A be a finite dimensional algebra over an algebraically closed field k , and let $\text{mod-}A$ be the category of finite dimensional right A -modules. For $M \in \text{mod-}A$ we denote by $\text{pd}_A M$ the projective dimension of M , and by $\text{add } M$ the full subcategory of direct sums of direct summands of M . Let $Q = (Q_0, Q_1)$ be a finite connected quiver without loops and cycles, and Q_0 (resp. Q_1) be the set of vertices (resp. arrows) of Q . For any $\alpha : x \rightarrow y$ set $s(\alpha) := x$ and $t(\alpha) := y$. And for any $x \in Q_0$ put $s(x) := \{\alpha \in Q_1 \mid s(\alpha) = x\}$ and $t(x) := \{\beta \in Q_1 \mid t(\beta) = x\}$ (we use these notations for an arbitrary quiver). We denote by kQ the path algebra of Q over k . A module $T \in \text{mod-}A$ is called a tilting module provided the following three conditions are satisfied:

- (a) $\text{pd } T < \infty$,
- (b) $\text{Ext}^i(T, T) = 0$ for all $i > 0$,
- (c) there exists an exact an sequence

$$0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_r \longrightarrow 0 \quad (T_i \in \text{add } T)$$

in $\text{mod-}A$. In the hereditary case the tilting condition above is equivalent to the following:

- (a) $\text{Ext}^1(T, T) = 0$,
- (b) the number of indecomposable direct summands of T (up to isomorphism) is equal to the number of simple modules.

We denote by $\vec{\mathcal{K}}(Q)$ the tilting quiver over Q . In this paper we will see a structure of $\vec{\mathcal{K}}_{p,p}(Q)$ the post-projective part of $\vec{\mathcal{K}}(Q)$ when Q satisfies some conditions (see Section 2). Now $\vec{\mathcal{K}}_{p,p}(Q)$ is the full subquiver of $\vec{\mathcal{K}}(Q)$ having post-projective basic tilting modules as the set of vertices.

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In Section 1, following [6], [7], [8], [9], we recall some definitions and properties used in this paper. In section 2 we give our main Theorem. In Section 3 we give Ext vanishing condition for post-projective modules. More precisely we give a function l_Q from $Q_0 \times Q_0$ to $\mathbb{Z}_{\geq 0}$ such that $\text{Ext}_{kQ}^1(\tau^{-r_i}P(i), \tau^{-r_j}P(j)) = 0$ if and only if $r_i \leq r_j + l_Q(j, i)$ and by using this fact we prove our main Theorem. In Section 4 we will see a structure of $\vec{\mathcal{K}}_{p,p}(Q)$ in the case $l(Q) := \max\{l_Q(x) \mid x \in Q_0\} \leq 1$.

In this paper we identify two quivers Q and Q' if Q is isomorphic to Q' as a quiver.

1. PRELIMINARIES

In this section we define a partial order on tilting modules. First, for a tilting module T , we define the right perpendicular category

$$T^\perp = \{X \in \text{mod-}A \mid \text{Ext}_A^{>0}(T, X) = 0\}.$$

Lemma 1.1. (cf. [7, lemma2.1 (a)]) *Let T and T' are two tilting modules. Then the following conditions are equivalent,*

- (1) : $T^\perp \subset T'^\perp$,
- (2) : $T \in T'^\perp$.

Recall that $\mathcal{T}(A)$ is the set of basic tilting modules of A .

Definition 1.2. We define a partial order on $\mathcal{T}(A)$ by

$$T \leq T' \stackrel{\text{def}}{\iff} T^\perp \subset T'^\perp \iff T \in T'^\perp,$$

for $T, T' \in \mathcal{T}(A)$.

Remark 1.3. By definition, A_A is the unique maximal element of $(\mathcal{T}(A), \leq)$. On the other hand, $(\mathcal{T}(A), \leq)$ does not always admit a minimal element (c.f [6]).

Next we define the *tilting quiver* $\vec{\mathcal{K}}(A)$, and recall its some properties. Let $\text{ind } A$ be a category of indecomposable modules in $\text{mod-}A$.

Definition 1.4. The *tilting quiver* $\vec{\mathcal{K}}(A) = (\vec{\mathcal{K}}(A)_0, \vec{\mathcal{K}}(A)_1)$ is defined as follows.

- (1) $\vec{\mathcal{K}}(A)_0 = \mathcal{T}(A)$,
- (2) $T' \rightarrow T$ in $\vec{\mathcal{K}}(A)$, for $T, T' \in \mathcal{T}(A)$, if $T' = M \oplus X$, $T = M \oplus Y$ with $X, Y \in \text{ind } A$ and there is a non-split short exact sequence

$$0 \longrightarrow X \longrightarrow \widetilde{M} \longrightarrow Y \longrightarrow 0$$

with $\widetilde{M} \in \text{add } M$.

Theorem 1.5. (cf. [6, thm 2.1]) *$\vec{\mathcal{K}}(A)$ is the Hasse-diagram of $(\text{Tilt}(A), \leq)$ (i.e. if $T \rightarrow T' \in \vec{\mathcal{K}}(A)_1$ and $T \geq T'' \geq T'$ then $T'' = T$ or $T'' = T'$).*

Proposition 1.6. (cf. [6, cor 2.2]) *If $\vec{\mathcal{K}}(A)$ has a finite component \mathcal{C} , then $\vec{\mathcal{K}}(A) = \mathcal{C}$.*

Let \mathcal{Q} be a set of finite connected quivers without loops and cycles.

Theorem 1.7. (cf.[8, thm 6.4]) *If $Q \in \mathcal{Q}$ has no multiple arrows, then Q is uniquely determined by $(\mathcal{T}(Q), \leq)$.*

Let M be a basic partial tilting module and $\text{lk}(M) := \{T \in \mathcal{T} \mid M \mid T\}$. Then we denote by $\vec{\text{lk}}(M)$ the full subquiver of $\vec{\mathcal{K}}(Q)$ having $\text{lk}(M)$ as a set of vertices (see [9]).

Proposition 1.8. (cf.[9]) *If M is faithful then $\vec{\text{lk}}(M)$ is connected.*

2. THEOREM

Let Q be a quiver with n vertices. For any $x \in Q_0$ we denote by $P(x)$ an indecomposable projective module associated with x . We denote by $\mathcal{T}_{p,p}(Q)$ a set of basic post-projective tilting modules, and for a basic post-projective partial tilting module M define $\text{lk}_{p,p}(M)$, $\vec{\text{lk}}_{p,p}(M)$ similarly.

Theorem 2.1. *Assume that Q satisfies the following conditions (a), (b)*

- (a) Q has an unique source $s \in Q_0$,
- (b) for any $x \in Q_0$, $\#s(x) + \#t(x) > 1$.

Then the followings are hold,

- (1) $\mathcal{T}_{p,p}(Q) = \coprod_{r \geq 0} \text{lk}_{p,p}(\tau^{-r}P(s))$,
- (2) $\tau^{-r} : \vec{\text{lk}}_{p,p}(P(s)) \simeq \vec{\text{lk}}_{p,p}(\tau^{-r}P(s))$,
- (3) $\#\text{lk}_{p,p}(P(s)) \leq 2^{n-1}$,
- (4) $\vec{\text{lk}}_{p,p}(P(s))$ is a connected quiver,
- (5) Let $T \in \text{lk}_{p,p}(\tau^{-r}P(s))$ and $T' \in \text{lk}_{p,p}(\tau^{-r-t}P(s))$. If there is an arrow $T \rightarrow T'$, then $t = 0$ or $t = 1$.
- (6) $\vec{\mathcal{K}}_{p,p}(Q)$ is connected.

3. A PROOF OF THEOREM 2.1

Let Q be a quiver with $Q_0 = \{0, 1, \dots, n-1\}$. Now we can assume that if $\exists \alpha : i \rightarrow j$ in Q then $i > j$. For $x \in \{0, 1, \dots, n-1\}$ and $r > 0$, put $Q(x + rn) := \tau^{-r}P(x)$.

Proposition 3.1. (cf.[2]) *Let $A = kQ$ be a path algebra, and $M, N \in \text{mod-}A$ be a non-injective right A -modules. Then,*

$$\text{Hom}_A(M, N) \simeq \text{Hom}_A(\tau^{-1}M, \tau^{-1}N).$$

Proposition 3.2. (cf.[1]) *(Auslander-Reiten duality) Let $A = kQ$ be a path algebra, and $M, N \in \text{mod-}A$. Then,*

$$D \text{Hom}_A(M, N) \simeq \text{Ext}_A^1(N, \tau M).$$

Proposition 3.3. (cf.[4]) *Let $A = kQ$ be a path algebra and $M \in \text{ind } A$. Then for any indecomposable non-projective module X and almost split sequence*

$$0 \rightarrow \tau X \rightarrow E \rightarrow X \rightarrow 0,$$

we get

$$\dim \operatorname{Hom}(M, \tau X) - \dim \operatorname{Hom}(M, E) + \dim \operatorname{Hom}(M, X) = \begin{cases} 1 & X \simeq M \\ 0 & \text{otherwise.} \end{cases}$$

Let $d_a(b) := \dim \operatorname{Ext}^1(Q(b), Q(a))$. If Q satisfies the condition (b) of Theorem 2.1, then kQ is representation infinite and post-projective part of its AR-quiver is $\mathbb{Z}_{\leq 0}Q$ (c.f.[2]). So from above proposition and AR-duality, we get the following,

$$d_a(x+rn) = \begin{cases} 0 & x+rn < a+n \\ 1 & x+rn = a+n \\ \sum_{\alpha \in s(x)} d_a(t(\alpha) + rn) + \sum_{\beta \in t(x)} d_a(s(\beta) + (r-1)n) - d_a(x + (r-1)n) & x+rn > a+n, \end{cases}$$

Lemma 3.4. *Assume Q satisfies the condition (b) of Theorem 2.1. Let $\gamma : x \rightarrow y$ in Q . Then,*

$$d_a(y+rn) \leq d_a(x+rn) \leq d_a(y+(r+1)n)$$

for any $r \geq 0$.

Proof. We can assume $a < n$, and use induction on $y+rn$.

($y+rn < n$ (i.e. $r = 0$)): In this case $d_a(y) = d_a(x) = 0$.

($n \leq y+rn < a+n$): In this case $d_a(y+rn) = 0$ and

$$d_a(y+(r+1)n) = \sum_{\alpha \in s(y)} d_a(t(\alpha) + (r+1)n) + \sum_{\beta \in t(y)} d_a(s(\beta) + rn) \geq d_a(x+rn).$$

($y+rn \geq a+n$): In this case

$$\begin{aligned} d_a(x+rn) &= \sum_{\alpha \in s(x)} d_a(t(\alpha)) + \sum_{\beta \in t(x)} d_a(s(\beta)) - d_a(x + (r-1)n) \\ &= \sum_{\alpha \in s(x) \setminus \{\gamma\}} d_a(t(\alpha) + rn) + \sum_{\beta \in t(x)} d_a(s(\beta) + (r-1)n) \\ &\quad - d_a(x + (r-1)n) + d_a(y+rn) \\ (*) \cdots &\geq d_a(y+rn) \end{aligned}$$

(remark. (*) is followed by (b) and hypothesis of induction.) and similarly we can get $d_a(y+(r+1)n) \geq d_a(x+rn)$. \square

Let \tilde{Q} be a quiver obtained from Q by adding new edge $-\alpha : y \rightarrow x$ for any $\alpha : x \rightarrow y$. For a path $w : x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_r} x_r$ in \tilde{Q} , put $c^+(w) := \#\{t \mid \alpha_t \in Q_1 \subset \tilde{Q}_1\}$, and let $l_Q(i, j) := \min\{c^+(w) \mid w : \text{path from } i \text{ to } j \text{ in } \tilde{Q}\}$ (set $l_Q(i, i) = 0$ for any i).

Proposition 3.5. *If Q satisfies the condition (b) of Theorem 2.1, then*

$$\operatorname{Ext}_{kQ}^1(\tau^{-r}P(i), \tau^{-s}P(j)) = 0 \Leftrightarrow r \leq s + l_Q(j, i)$$

Proof. (\Rightarrow): Let $w : j = x_0 \xrightarrow{\alpha_1} x_1 \rightarrow \cdots \xrightarrow{\alpha_t} x_t = i$ be a path s.t. $l(j, i) := l_Q(j, i) = c^+(w)$ and $\{k_1 < k_2 < \cdots < k_{l(j, i)}\} = \{k \mid \alpha_k \in Q_1\}$. If $\exists r > l(j, i)$ s.t. $\operatorname{Ext}_{kQ}^1(\tau^{-r}P(i), P(j)) = 0$, then, by Lemma 3.4, we get

$$\begin{aligned} 0 &= d_j(x(t) + rn) \geq d_j(x_{k_{l(j, i)}} + (r-1)n) \geq \cdots \geq d_j(x_{k_1} + (r-l(j, i))n) \\ &\geq d_j(j + (r-l(j, i))n) \geq d_j(j + (r-l(j, i) - 1)n) \geq \cdots \geq d_j(j+n) > 0, \end{aligned}$$

and this is a contradiction. So if $\text{Ext}_{kQ}^1(\tau^{-r}P(i), \tau^{-s}P(j)) = 0$ with $r \leq s + l(j, i)$, then by proposition 3.1, we get a contradiction.

(\Leftarrow) Let $\mathcal{A}(j) := \{(i, r) \mid r \leq l(j, i), \text{Ext}_{kQ}^1(\tau^{-r}P(i), P(j)) \neq 0\}$. If $\mathcal{A}(j) \neq \emptyset$, then we can take $r := \min\{r \mid (i, r) \in \mathcal{A}(j) \text{ for some } i\}$ and $i \in Q_0$ s.t. $(i', r) \notin \mathcal{A}(j)$ for any $i' \leftarrow i$ in Q . Now

$$0 < d_j(i + rn) \leq \sum_{\alpha \in s(i)} d_j(t(\alpha) + rn) + \sum_{\beta \in t(i)} d_j(s(\beta) + (r-1)n)$$

implies that $d_j(t(\alpha) + rn) \neq 0$ for some $\alpha \in s(i)$ or $d_j(s(\beta) + (r-1)n) \neq 0$ for some $\beta \in t(i)$. Note that $r \leq l(j, i) \leq l(j, t(\alpha)), l(j, s(\beta)) + 1$ for any $\alpha \in s(i)$ and $\beta \in t(i)$. So $d_j(t(\alpha) + rn) = 0 = d_j(s(\beta) + (r-1)n)$ for any $\alpha \in s(i)$ and $\beta \in t(i)$ and this is a contradiction. So we get $\mathcal{A}(j) = \emptyset$.

We assume that $\exists i \in Q_0$ and $\exists r, s \in \mathbb{Z}_{\geq 0}$ s.t. $r \leq s + l(j, i)$ and

$$\text{Ext}_{kQ}^1(\tau^{-r}P(i), \tau^{-s}P(j)) \neq 0.$$

If $r < s$, then proposition 3.1 shows $\text{Ext}_{kQ}^1(\tau^{-r}P(i), \tau^{-s}P(j)) \neq 0$. So $r \geq s$. Now proposition 3.1 implies $(i, r-s) \in \mathcal{A}(j)$ and this is a contradiction. \square

Lemma 3.6. *Let $T \in \mathcal{T}_{p,p}(Q)$ and $T \leq T' \in \mathcal{T}(Q)$. Then $T' \in \mathcal{T}_{p,p}(Q)$. In particular $\vec{\mathcal{K}}_{p,p}(Q)$ (resp. $\vec{\text{lk}}_{p,p}(M)$) is a Hasse-diagram of $(\mathcal{T}_{p,p}, \leq)$ (resp. $\text{lk}_{p,p}(M), \leq$).*

Proof. Let X be an indecomposable direct summand of T' . If X is not post-projective, then $\text{Ext}^1(\tau^{-r}P, X) \simeq \text{Ext}^1(P, \tau^r X) = 0$ for any projective module P . So $\text{Ext}^1(T, X) = 0$. Since $\text{Ext}^1(X, T) = 0$, we get $X \mid T$. This is a contradiction. \square

For any quiver Q satisfying the condition (b) of Theorem 2.1, put $L(Q) := \{(r_i)_{i \in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0} \mid r_j \leq r_i + l_Q(i, j)\} \subset \mathbb{Z}^{Q_0}$.

Then as an immediate corollary of proposition 3.5, we get the following.

Corollary 3.7. *Assume Q satisfies the conditions (b) of Theorem 2.1. Then*

$$(r_i)_{i \in Q_0} \mapsto \bigoplus_{i=0}^{n-1} \tau^{-r_i} P(i)$$

induces an isomorphism of posets,

$$(L(Q), \leq^{op}) \simeq (\mathcal{T}_{p,p}(Q), \leq)$$

where $(r_i)_{i \in Q_0} \geq^{op} (r'_i)_{i \in Q_0} \stackrel{\text{def}}{\iff} r_i \leq r'_i$ for any $i \in Q_0$.

Proposition 3.8. *Let $i \in Q_0$ and $T(i) := \bigoplus_{j \in Q_0} \tau^{-l(i,j)} P(j)$. Then $T(i)$ is a unique minimal element of $\text{lk}_{p,p}(P(i))$.*

Proof. Let $j, j' \in Q_0$. By definition of l_Q , we get

$$l(i, j) \leq l(i, j') + l(j', j),$$

and this implies $T(i) \in \text{lk}_{p,p}(P(i))$.

Now let $T := \oplus_{j \in Q_0} \tau^{-r_j} P(j) \in \text{lk}_{p,p}(P(i))$. Then $r_j \leq r_i + l(i, j) = l(i, j)$. So corollary 3.7 shows $T \geq T(i)$. \square

Now we can show the Theorem 2.1.

Proof. (1): This is followed by proposition 3.5.

(2): It is obvious that τ^{-r} induces an injection

$$\vec{\text{lk}}_{p,p}(P(n-1)) \rightarrow \vec{\text{lk}}_{p,p}(\tau^{-r} P(n-1))$$

as a quiver. So it is sufficient to show

$$\tau^{-r} : \text{lk}_{p,p}(P(n-1)) \rightarrow \text{lk}_{p,p}(\tau^{-r} P(n-1))$$

is surjective.

Let $T \in \text{lk}_{p,p}(\tau^{-r} P(n-1))$ and by corollary 3.7 we can put $T = \oplus_{i=0}^{n-2} \tau^{-r_i} P(i) \oplus \tau^{-r} P(n-1)$. Since $l(i, n-1) = 0$ ($\forall i$), proposition 3.5 shows $r_i \geq r$ ($\forall i$). So $\tau^r T \in \text{lk}_{p,p}(P(n-1))$

(3): For any $i \in Q_0$ put $t(i) := \max\{j \in Q_0 \mid j \rightarrow i\}$. Let $T = \oplus_{i=0}^{n-1} \tau^{-r_i} P(i)$. Then $T \in \text{lk}_{p,p}(P(n-1))$ only if

$$r_{n-1} = 0, r_{n-2} \in \{0, 1\}, \dots, r_i \in \{r_{t(i)}, r_{t(i)} + 1\}, \dots, r_0 \in \{r_{t(0)}, r_{t(0)} + 1\}.$$

This implies $\#Q_0 \leq 2^{n-1}$.

(4): Let \mathcal{C} be a connected component of $\vec{\text{lk}}_{p,p}(P(n-1))$ containing $A = kQ$. If $T \in \mathcal{T}_{p,p} \setminus \mathcal{C}$, then there is an infinite sequence

$$A = T_0 \rightarrow T_1 \rightarrow T_2 \rightarrow \dots$$

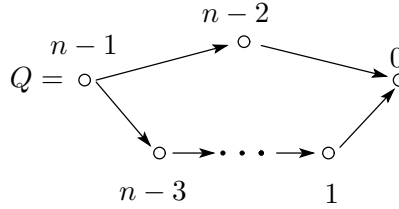
in \mathcal{C} (cf.[7]), and this is a contradiction.

(5): Let $T = \oplus_{i \in Q_0} \tau^{-r_i} P(i) \in \text{lk}_{p,p}(\tau^{-r} P(n-1))$ and $T' = \oplus_{i \in Q_0} \tau^{-r'_i} P(i) \in \text{lk}_{p,p}(\tau^{-r+t} P(n-1))$. If $T \rightarrow T'$ in $\vec{\mathcal{K}}_{p,p}(Q)$ (i.e. in $\vec{\mathcal{K}}(Q)$) and $t \neq 0$, then $r_i = r'_i$ ($\forall i \in Q_0 \setminus \{n-1\}$), $r_{n-1} \leq r'_{n-1} = r_{n-1} + t$ and $r_{n-1} + t = r'_{n-1} \leq r'_{n-2} = r_{n-2} \leq r_{n-1} + l(n-1, n-2) = r_{n-1} + 1$. This implies $t = 0$.

(6): Put $T := (\oplus_{i \leq n-2} \tau^{-r} P(i)) \oplus \tau^{-r+1} P(n-1)$ and $T' := \oplus_{i \leq n-1} \tau^{-r} P(i)$. Then corollary 3.7 shows $T \in \text{lk}_{p,p}(\tau^{-r+1} P(n-1))$ and $T' \in \text{lk}_{p,p}(\tau^{-r} P(n-1))$ with $T \rightarrow T'$. Now (4) of this theorem implies $\vec{\mathcal{K}}_{p,p}(Q)$ is connected. \square

Example 3.9. We give the two examples. Recall that for any two elements $(r_i), (r'_i)$ of \mathbb{Z}^n ($n \geq 1$), $(r_i) \geq^{op} (r'_i)$ means $r_i \leq r'_i$ for any i .

(1): Consider the following quiver



Let

$$\begin{aligned} L_0 &= \{(0, 0, 0 \dots 0, 0, 0), (1, 0, 0 \dots 0, 0, 0), (1, 1, 0 \dots 0, 0, 0), \dots, (1, 1, 1 \dots 1, 0, 0)\} \\ L_1 &= \{(1, 0, 0 \dots 0, 1, 0), (1, 1, 0 \dots 0, 1, 0), (1, 1, 1 \dots 0, 1, 0), \dots, (1, 1, 1 \dots 1, 1, 0)\} \end{aligned}$$

$$L_2 = \{(2, 1, 0 \cdots 0, 1, 0), (2, 1, 1 \cdots 0, 1, 0), \cdots, (2, 1, 1 \cdots 1, 1, 0)\}$$

$$L_3 = \{(2, 2, 1, 0 \cdots 0, 1, 0), \cdots, (2, 2, 1 \cdots 1, 1, 0)\}$$

$$\vdots$$

$$L_{n-3} = \{(2, 2, 2 \cdots 2, 1, 1, 0)\}$$

and

$$T(a, b) = \begin{cases} \text{b-th element of } L_a & 0 \leq a \leq n-3, 1 \leq b \leq n-1-a \\ T(b-n+a, n-b) + (1, \cdots, 1) & 0 \leq a \leq n-3, n-1-a < b \leq n-1 \\ T(x, b) + (2r, 2r, \cdots, 2r) & a = x + (n-2)r \ (0 \leq x < n-2), 1 \leq b \leq n-1. \end{cases}$$

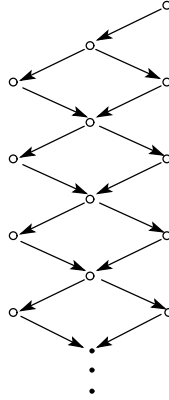
Now we get

$$(\coprod L_a, \leq^{op}) \simeq (\text{lk}_{p,p}(P(n-1)), \leq)$$

and

$$(L(Q), \leq^{op}) = (\{T(a, b) \mid a \in \mathbb{Z}_{\geq 0}, 1 \leq b \leq n-1\}, \leq^{op}) \simeq (\mathbb{Z}_{\geq 0} \times \{1, \cdots, n-1\}, \leq^{op}).$$

In particular $\vec{\mathcal{K}}_{p,p}(Q) = \mathbb{Z}_{\geq 0} \vec{A}_{n-1}$. So, in the case $n = 4$, $\vec{\mathcal{K}}_{p,p}(Q)$ is given by following



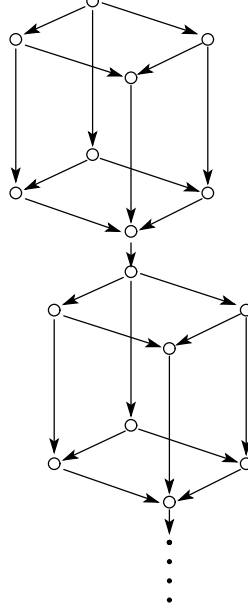
(2): Consider the following quiver

$$Q = \begin{array}{ccc} & n-1 & \circ \quad 0 \\ & \nearrow & \nearrow \\ \circ & & \vdots \\ & \searrow & \searrow \\ & n-2 & \circ \end{array}$$

It is easy to see that

$$(\{0, 1\}^{n-1}, \leq^{op}) \simeq (\text{lk}_{p,p}(P(n-1)), \leq)$$

and so an underlying graph of $\vec{\text{lk}}_{p,p}(P(n-1))$ is isomorphic to $(n-1)$ -dimensional cube. In the case $n = 4$, $\vec{\mathcal{K}}_{p,p}(Q)$ is given by following,



4. AN APPLICATION

In this section we consider a quiver $\vec{\mathcal{K}}_{p,p}(Q)$ for Q satisfying condition (b) of Theorem 2.1 and $l(Q) := \max\{l_Q(x) \mid x \in Q_0\} \leq 1$.

For any quiver $Q \in \mathcal{Q}$ define a new quiver Q° by adding new edges $x \rightarrow y$ for any source x which is not sink and sink y which is not source. Then let $\mathcal{A} := \{Q \in \mathcal{Q} \mid Q \text{ has a unique source}\}$, $\mathcal{B} := \{Q \in \mathcal{A} \mid Q \text{ has a unique sink}\}$ and $\mathcal{A}^\circ := \{Q^\circ \mid Q \in \mathcal{A}\}$. Note that $\mathcal{A}^\circ = \{Q \in \mathcal{A} \mid l(Q) \leq 1\}$.

Definition 4.1. we define the maps $\vec{\Pi} : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathcal{Q}$, $\phi : \mathcal{A}^\circ \times \mathcal{A}^\circ \rightarrow \mathcal{A}^\circ$, $\psi : \mathcal{A}^\circ \rightarrow \mathcal{B}$ and $\Psi : \mathcal{A}^\circ \times \mathcal{A}^\circ \rightarrow \mathcal{B} \times \mathcal{B}$ as follows,

- (1) $(Q \vec{\Pi} Q')_0 := Q_0 \amalg Q'_0$,
 $(Q \vec{\Pi} Q')_1 := Q_1 \amalg Q'_1 \amalg \{y \rightarrow x' \mid y : \text{source of } Q, x' : \text{sink of } Q'\}$,
- (2) $\phi(Q, Q') := (Q \vec{\Pi} Q')^\circ$,
- (3) $\psi(Q) := \vec{\text{lk}}_{p,p}(P_Q)$, where P_Q is an indecomposable projective module associated with a unique source,
- (4) $\Psi(Q, Q') := (\psi(Q'), \psi(Q))$.

Proposition 4.2. *The following diagram is commutative,*

$$\begin{array}{ccc}
 \mathcal{A}^\circ \times \mathcal{A}^\circ & \xrightarrow{\Psi} & \mathcal{B} \times \mathcal{B} \\
 \downarrow \phi & & \downarrow \vec{\Pi} \\
 \mathcal{A}^\circ & \xrightarrow{\psi} & \mathcal{B}
 \end{array}$$

Proof. Let $Q(1), Q(2) \in \mathcal{A}^\circ$, $s(k)$ be a unique source of $Q(k)$ ($k = 1, 2$) and $T = \oplus_{i \in \phi(Q(1), Q(2))_0} \tau^{-r_i} P(i)$. Then corollary 3.7 shows $T \in \psi(\phi(Q(1), Q(2)))_0$ if and only if (i): $(r_i)_{i \in Q(2)_0} \in L(Q(2))$ with $r_{s(2)} = 0$ and $r_j = 0$ ($\forall j \in Q(1)_0$) or (ii): $(r_j)_{j \in Q(1)_0} \in L(Q(1))$ with $r_{s(1)} = 0$ and $r_i = 1$ ($\forall i \in Q(2)_0$). Now it is easy to check that

$$\psi(\phi(Q(1), Q(2))) = \psi(Q(2))\vec{\Pi}\psi(Q(1)).$$

□

Now we define a relation \rightsquigarrow on \mathcal{A}° as follows,

$$Q \rightsquigarrow Q' \stackrel{\text{def}}{\iff} Q' = Q \setminus \{\alpha\}$$

for some $\alpha \in Q_1$ satisfying the conditions (1) or (2);

(1) $s(\alpha)$ is not source or $t(\alpha)$ is not sink and $\exists w \neq \alpha$ path from $s(\alpha)$ to $t(\alpha)$.

(2) $s(\alpha)$ is a source, $t(\alpha)$ is a sink and there is at least three paths from $s(\alpha)$ to $t(\alpha)$ and at least two arrows from $s(\alpha)$ to $t(\alpha)$.

Let $\mathcal{S} := \{Q \in \mathcal{A}^\circ \mid \text{there is a no quiver } Q' \text{ s.t. } Q \rightsquigarrow Q'\}$. We can easily see that if

$$Q \rightsquigarrow \dots \rightsquigarrow Q' \in \mathcal{S}, \quad Q \rightsquigarrow \dots \rightsquigarrow Q'' \in \mathcal{S},$$

then $Q' = Q''$ and in this case put $\pi(Q) := Q' = Q''$. Now we define a equivalence relation \sim on \mathcal{A}° as follows,

$$Q \sim Q' \stackrel{\text{def}}{\iff} \pi(Q) = \pi(Q').$$

Lemma 4.3. *Let $Q, Q' \in \mathcal{A}^\circ$. If $Q \sim Q'$, then $\psi(Q) = \psi(Q')$. In particular we get a map*

$$\psi / \sim: \mathcal{A}^\circ / \sim \rightarrow \mathcal{B}.$$

Proof. Note that $l_Q(i, j) = 0$ if and only if there exists a path from j to i . So if $Q \rightsquigarrow Q'$ then $l_Q(i, j) = l_{Q'}(i, j)$ for any $i, j \in Q_0 = Q'_0$. In particular corollary 3.7 shows $\psi(Q) = \psi(Q')$. □

Lemma 4.4. *Let $Q(1), Q(2), Q'(1), Q'(2) \in \mathcal{A}^\circ$. If $Q(i) \sim Q'(i)$ ($i = 1, 2$), then $\phi(Q(1), Q(2)) \sim \phi(Q'(1), Q'(2))$. In particular we get a map*

$$\phi / \sim: \mathcal{A}^\circ / \sim \times \mathcal{A}^\circ / \sim \rightarrow \mathcal{A}^\circ / \sim.$$

Proof. Let $Q, Q', Q'' \in \mathcal{A}^\circ$ with $Q \rightsquigarrow Q' = Q \setminus \{\alpha\}$. By definition we get $\phi(Q, Q'') = \phi(Q, Q'') \setminus \{\alpha\}$. And now α satisfies the condition (1) of definition for $\phi(Q', Q'') \rightsquigarrow \phi(Q', Q'')$. In particular we get

$$\phi(Q, Q'') \rightsquigarrow \phi(\pi(Q), Q'').$$

Similarly we can see that

$$\phi(Q, Q'') \rightsquigarrow \phi(Q, \pi(Q'')).$$

So we get $\phi(Q, Q'') \rightsquigarrow \phi(\pi(Q), \pi(Q''))$. \square

Let C^n be a Hasse-diagram of $(\{0, 1\}^n, \leq)$, and $Q \in \mathcal{A}^\circ$ with $Q_0 = \{s, 1, 2, \dots, n-1\}$ where s is a unique source of Q . Then a map

$$\rho : P(s) \oplus (\oplus_{i=1}^{n-1} \tau^{-r_i} P(i)) \mapsto (r(i))_i$$

induces an injection $\psi(Q) \rightarrow C^{n-1}$ as a quiver. So we identify $\psi(Q)$ as a full sub-quiver of C^{n-1} . Note that $(0, \dots, 0), (1, \dots, 1) \in \psi(Q)$. Now for any $T \in C_0^{n-1}$ denote by T_i the i -th entry of T .

Proposition 4.5. *Let $Q \in \mathcal{A}^\circ$. Then $\psi(Q) = K(1) \vec{\Pi} K(2)$ for some quivers $K(1), K(2) \in \mathcal{B}$ if and only if $\exists (Q(1), Q(2)) \in \mathcal{A}^\circ \times \mathcal{A}^\circ$ s.t. $\psi(Q(i)) = K(i)$ ($i = 1, 2$) and $\phi(Q(2), Q(1)) \sim Q$.*

Proof. By above lemma we can assume that $Q \in \mathcal{S}$. Let T be the unique minimal element of K_1 and T' be the unique maximal element of K_2 . Then $\exists ! i$ s.t. $T_i = 0, T'_i = 1$. Note that $T'' \leq T$ or $T'' > T$ for any $T'' \in \psi(Q)$ (note also that $T'' > T'$ or $T'' \leq T'$ for any $T'' \in \psi(Q)$). Since $T' < T(i) \leq T$ we get $T = T(i)$. If $T(j) \leq T = T(i)$ then $l(i, j) \leq l(j, j) = 0$, and if $T(j) > T(i)$ then $l(j, i) \leq l(i, i) = 0$. So for any $j \leq n-1$ there is a path from i to j or a path from j to i . This implies that $Q = (Q'(2) \vec{\Pi} Q'(1))^\circ$, where $Q'(1) \in \mathcal{A}$ (resp. $Q'(2)$) is the full sub-quiver of Q with $Q'(1)_0 = \{j \mid j \neq i, l(j, i) = 0\}$ (resp. $Q'(2)_0 = \{j \mid l(i, j) = 0\}$) (remark. we use the fact $Q \in \mathcal{S}$). Let $Q(i) := Q'(i)^\circ$, then $(Q'(2) \vec{\Pi} Q'(1))^\circ \sim \phi(Q(2), Q(1))$.

Now it is sufficient to show that $\psi(Q(i)) = K(i)$. Consider a injection

$$\iota : \psi(Q(1)) \rightarrow K(1),$$

$$\text{where } \iota(T'')_j := \begin{cases} T''_j & j \in Q(1)_0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $T'' \in K(1)$, then $T''_j \leq T''_{j'} + l_Q(j', j) = T''_{j'} + l_{Q(1)}(j', j)$ for any $j, j' \in Q(1)_0$. Since $T'' \geq T(i)$ we get $T''_j \leq T(i)_j = l_Q(i, j) = 0$ for any $j \in Q(2)_0$ and this implies $T'' \in \iota(\psi(Q(1)))$. So we get $\psi(Q(1)) = K(1)$.

Similarly we can see that $\psi(Q(2)) = K(2)$. \square

Now for any $K \in \mathcal{B}$ with the following properties,

- (1) K is a full sub-quiver of C^{n-1} for some n and $(0, \dots, 0), (1, \dots, 1) \in K_0$,
- (2) for any $T > T'$ in K , there is a path from T to T' in K ,

we define $Q(K) \in \mathcal{S}$ as follows. Let $Q'(K)$ be a Hasse-diagram of the poset $(\{1, 2, \dots, n-1\}, \leq_K)$, where $i \leq_K j \stackrel{\text{def}}{\iff} T_i \geq T_j$ for any $T \in K$ (Remark. Assume $T_i = T_j$ for any $T \in K$. Now there is a path $(0, \dots, 0) = T^0 \rightarrow T^1 \rightarrow \dots \rightarrow T^r = (1, \dots, 1)$ in K and so $\exists a$ s.t. $T_i^{a-1} = T_j^{a-1} = 0$ and $T_i^a = T_j^a = 1$. This implies $i = j$). And set $Q(K) := (\{\circ\} \vec{\Pi} Q'(K))^\circ$.

Lemma 4.6. *We get $Q \sim Q(\psi(Q))$ for any $Q \in \mathcal{A}^\circ$. In particular ψ/\sim is injective.*

Consider the following condition (3) of K satisfying (1) and (2),
 (3) $\langle T, T' \rangle_\pm \in K_0$ for any $T, T' \in K_0$, where $\langle T, T' \rangle_+ := (\min\{T_i, T'_i\})_i$
 and $\langle T, T' \rangle_- := (\max\{T_i, T'_i\})_i$.

Let $\mathcal{L} := \{K \in \mathcal{B} \mid K \text{ satisfies (1), (2), (3)}\}$

Lemma 4.7. *Let $K \in \mathcal{B}$. Then the followings are equivalent,*

- (1) $K = \psi(Q)$ for some $Q \in \mathcal{A}^\circ$,
- (2) $K \in \mathcal{L}$.

Proof. ((1) \Rightarrow (2)): Let $K = \psi(Q)$ with $Q \in \mathcal{A}^\circ$ and $n := \#Q_0$. Then we have already seen that K is a full sub-quiver of C^{n-1} and $(0, \dots, 0), (1, \dots, 1) \in K_0$. So K satisfies the condition (1) for \mathcal{L} . Note that K also satisfies the condition (2) for \mathcal{L} (see the proof of theorem 2.1 (3)).

Now it is sufficient to prove that K satisfies the condition (3) for \mathcal{L} . Let $T, T' \in K_0$ and $i, j \in Q_0$. If $\min\{T_i, T'_i\} = 0$ or $l_Q(j, i) = 1$, then $\min\{T_i, T'_i\} \leq \min\{T_j, T'_j\} + l_Q(j, i)$. Assume that $\min\{T_i, T'_i\} = 1$ and $l_Q(j, i) = 0$. In this case $1 = T_i \leq T_j$ and $1 = T'_i \leq T'_j$ and this implies $T_j = T'_j = 1$. In particular $\min\{T_i, T'_i\} \leq \min\{T_j, T'_j\} + l_Q(j, i)$ for any $i, j \in Q_0$. So $\langle T, T' \rangle_+ \in K_0$. Similarly we can show that $\langle T, T' \rangle_- \in K_0$.

((2) \Rightarrow (1)): Let $K \in \mathcal{L}$. We will show that $K = \psi(Q(K))$.

First let $T \in K$ and $i, j \in Q(K)_0$. If $l(j, i) = l_{Q(K)}(j, i) = 0$ then $j \leq_K i$ and this implies $T_i \leq T_j$. If $l(j, i) = 1$ then $T_i \leq T_j + l(j, i)$. So $T \in \psi(Q(K))$.

Next assume that $\psi(Q(K)) \setminus K \neq \emptyset$ and let T be a minimal element of $\{T \in \psi(Q(K)) \setminus K \mid T' \rightarrow T \text{ for some } T' \in K\}$. Let $T' \in K$ with $T' \rightarrow T$, then the conditions (1) and (2) implies that $\exists T'' \in K$ s.t. $T' \rightarrow T''$. Now $\exists i, j$ s.t. $T_i = 1, T'_i = T'_j = 0$ and $T''_j = 1$. By lemma 4.6 $\leq_K = \leq_{\psi(Q(K))}$ and so $T_i > T_j$ implies $\exists S \in K$ s.t. $S_j < S_i$. Let $T''' := \langle T, T'' \rangle_-$, then minimality of T implies $T''' \in K$. Now

$$(\langle \langle T', S \rangle_-, T''' \rangle_+)_a = \begin{cases} \min\{\max\{T_a, S_a\}, T_a\} & a \neq i, j \\ 1 & a = i \\ 0 & a = j. \end{cases}$$

So $T = \langle \langle T', S \rangle_-, T''' \rangle_+ \in K$. This is a contradiction. □

Corollary 4.8. *ψ induces a bijection between \mathcal{S} and \mathcal{L} .*

Now, by applying theorem 2.1, we get the following result.

Theorem 4.9. (1) For any $Q \in \mathcal{A}^\circ$, there exists $K \in \mathcal{L}$ s.t. $\vec{\mathcal{K}}_{p,p}(Q) = \vec{\Pi}K$.
 (2) For any $K \in \mathcal{L}$, there exists $Q \in \mathcal{A}^\circ$ s.t. $\vec{\mathcal{K}}_{p,p}(Q) = \vec{\Pi}K$.

Corollary 4.10. Let $Q(1), Q(2) \in \mathcal{A}^\circ$. Then the followings are equivalent,

- (1) $\vec{\mathcal{K}}_{p,p}(Q(1)) = \vec{\mathcal{K}}_{p,p}(Q(2))$,
- (2) $\exists Q \in \mathcal{A}^\circ$ s.t. $Q(1) \sim (Q\vec{\Pi}Q\vec{\Pi} \cdots \vec{\Pi}Q)^\circ$ and $Q(2) \sim (Q\vec{\Pi}Q\vec{\Pi} \cdots \vec{\Pi}Q)^\circ$.

Proof. ((2) \Rightarrow (1)): This is followed by proposition 4.5, lemma 4.3 and lemma 4.4.

((1) \Rightarrow (2)): Let

$$\psi(Q(i)) = S^1(i)\vec{\Pi}S^2(i)\vec{\Pi} \cdots \vec{\Pi}S^{r_i}(i) \quad (S^t(i) \in \mathcal{B})$$

be a decomposition with r_i being maximal ($i = 1, 2$) and $r := \gcd(r_1, r_2)$.

Consider a homomorphism $f : \mathbb{Z} \rightarrow \mathbb{Z}/r_1\mathbb{Z} \oplus \mathbb{Z}/r_2\mathbb{Z}$ where $f(t) = (t \bmod r_1, t \bmod r_2)$. Let $1 \leq a \leq r_1$ and $1 \leq b \leq r_2$. Then the condition (1) implies

$$a \equiv b \bmod r \Rightarrow (a \bmod r_1, b \bmod r_2) \in \text{Im } f \Rightarrow S^a(1) = S^b(2)$$

. So $S^{x+tr}(1) = S^x(2) = S^x(1)$ and $S^{x+tr}(2) = S^x(1) = S^x(2)$ ($x \leq t$). In particular we get

$$\psi(Q(1)) = S\vec{\Pi}S \cdots \vec{\Pi}S, \psi(Q(2)) = S\vec{\Pi}S \cdots \vec{\Pi}S,$$

where $S = S^1(1)\vec{\Pi}S^2(1)\vec{\Pi} \cdots \vec{\Pi}S^r(1)$. By proposition 4.5, we can chose Q satisfying $\psi(Q) = S$. Now lemma 4.6 shows Q satisfies the condition (2). \square

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